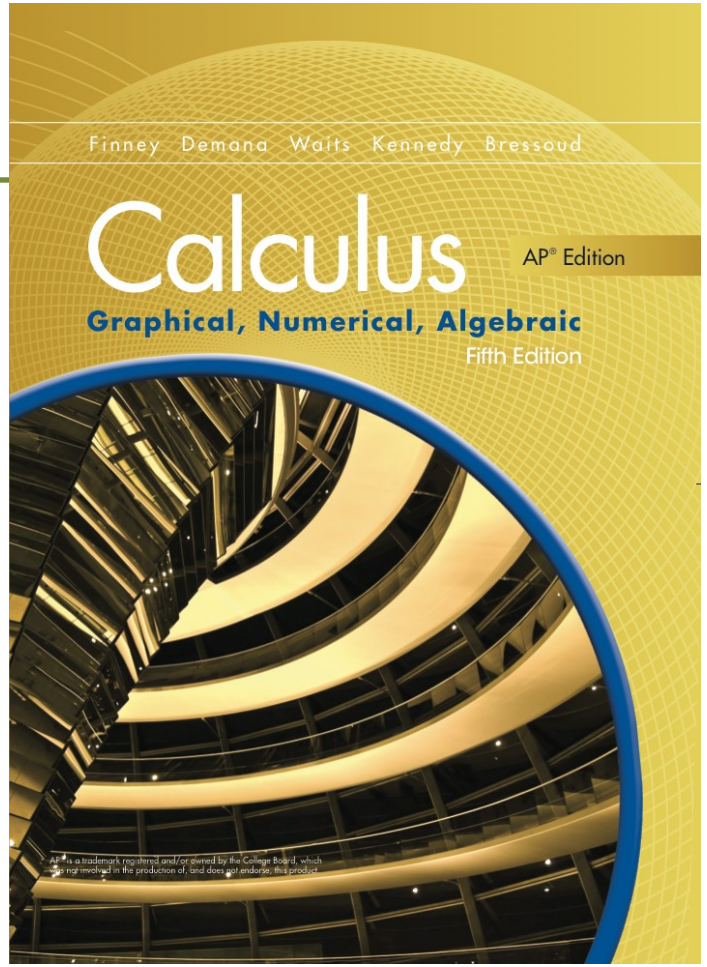
Chapter 8
Applications of Definite Integral

Section 8.5

Applications from Science and Statistics



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Quick Review

Find the definite integral by **(a)** antiderivatives and **(b)** using NINT.

1. $\int_1^1 e^{-x}\,dx$
2. $\int_0^{\frac{\pi}{3}} \cos x\,dx$
3. $\int_0^3 (x^4 - x)\,dx$
4. $\int_1^3 \frac{3x^2}{x^3+1}\,dx$

Find, but do not evaluate, the definite integral that is the limit as the norms of the partitions go to zero of the Reimann sums on the closed interval $[0,7]$.

5. $\sum 2\pi(x_k - 1)\sin x_k \Delta x$
6. $\sum 2\pi x_k^2 (2 + x_k)\,\Delta x$
7. $\sum \frac{\sqrt{3}}{4}(5 - y_k)\,\Delta y$

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Quick Review Solutions

Find the definite integral by **(a)** antiderivatives and **(b)** using NINT.

1. $\int_1^1 e^{-x}\,dx$ **a.** $e-\frac{1}{e}$ **b.** 2.350

2. $\int_0^{\frac{\pi}{3}} \cos x\,dx$ **a.** $\frac{\sqrt{3}}{2}$ **b.** 0.866

3. $\int_0^3 (x^4 - x)\,dx$ **a.** $\frac{441}{10}$ **b.** 44.1

4. $\int_1^3 \frac{3x^2}{x^3+1}\,dx$ **a.** $\ln(14)$ **b.** 2.639

Find, but do not evaluate, the definite integral that is the limit as the norms of the partitions go to zero of the Reimann sums on the closed interval $[0,7]$.

5. $\sum 2\pi(x_k - 1)\sin x_k \Delta x$ $\int_0^7 (2\pi(x-1)\sin x)\,dx$

6. $\sum 2\pi x_k^2(2+x_k)\Delta x$ $\int_0^7 (2\pi x^2(2+x))\,dx$

7. $\sum \frac{\sqrt{3}}{4}(5-y_k)\Delta y$ $\int_0^7 \left(\frac{\sqrt{3}}{4}(5-y)\right)dy$

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What you'll learn about

- Work done by variable force
- Total fluid pressure
- Normal probability distribution

...and why

It is important to see applications of integrals as

various accumulation functions

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Example Finding the Work Done by a Force

Find the work done by the force $F(x) = \sin(\pi x)$ newtons along the x-axis from $x = 0$ meters to $x = \frac{1}{2}$ meter.

$$W = \int_0^{\frac{1}{2}} \sin(\pi x)\, dx$$

$$= -\frac{1}{\pi}\cos(\pi x)\Big]_0^{\frac{1}{2}}$$

$$= -\frac{1}{\pi}(0 - 1)$$

$$= \frac{1}{\pi}$$

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Example Work Done Lifting

A leaky bucket weighs 15 newtons (N) empty. It is lifted from the ground at a constant rate to a point 10 m above the ground by a rope weighing 0.5 N/m. The bucket starts with 70 N of water, but it leaks at a constant rate and just finishes draining as the bucket reaches the top. Find the amount of work done

(a) lifting the bucket alone;

(b) lifting the water alone;

(c) lifting the rope alone;

(d) lifting the bucket, water, and rope together.

(a) Since the bucket's weight is constant, you must exert a force of 15 N through the entire 10-meter interval.

$W = (15\ \text{N}) \times (10\ \text{m}) = 150\ \text{N} \cdot \text{m} = 150\ \text{J}$

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Example Work Done Lifting

(b) The force needed to lift the water is equal to the water's weight, which decreases steadily from 70 N to 0 N over the 10-m lift. When the bucket is x meters off the ground, it weighs: $F(x) = \left(\begin{array}{l}\text{original weight}\\ \text{of water}\end{array}\right)\left(\begin{array}{l}\text{proportion left at}\\ \text{elevation } x\end{array}\right)$

$$F(x) = (70)\left(\frac{10 - x}{10}\right) = 70 - 7x \text{ N}.$$

The work done is

$$W = \int_0^{10} (70 - 7x)\,dx = \left[70x - \frac{7x^2}{2}\right]_0^{10} = 350 \text{ J}.$$

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Example Work Done Lifting

(c) The force needed to lift the rope also varies, starting at (0.5)(10)=5 N when the bucket is on the ground and 0 N when the bucket is is at the top. The rate of decrease is constant. At elevation x meters, the $(10 - x)$ meters of rope still there to lift weigh $F(x) = 0.5(10 - x)$ N.

$$W = \int_0^{10} 0.5(10 - x)\,dx$$

$$= \left[5x - 0.25x^2 \right]_0^{10}$$

$$= 25 \text{ J}.$$

(d) The total work is the sum: 25+350+150=525 J

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Fluid Force and Fluid Pressure

In any liquid, the **fluid pressure** p (force per unit area) at depth h is $p = wh$, where w is the weight-density (weight per unit volume) of the liquid.

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Example The Great Molasses Flood of 1919

At 1:00 pm on January 15, 1919, a 90-ft-high, 90-foot-diameter cylindrical metal tank in which the Puritan Distilling Company stored molasses at the corner of Foster and Commercial streets in Boston's North End exploded. Molasses flooded the streets
30 feet deep, trapping pedestrians and horses, knocking down buildings and oozing into homes. It was eventually tracked all over town and even made its way into the suburbs via trolley cars and people's shoes. It took weeks to clean up.

(a) Given that the tank was full of molasses weighing 100 lb/ft^3, what was the total force exerted by the molasses on the bottom of the tank at the time it ruptured?

(b) What was the total force against the bottom foot-wide band of the tank wall?

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Example The Great Molasses Flood of 1919

(a) At the bottom of the tank, the molasses exerted a constant pressure of $p = wh = \left(100 \frac{\text{lb}}{\text{ft}^3}\right)(90 \text{ ft}) = 9000 \frac{\text{lb}}{\text{ft}^2}$.

The area of the base was $\pi (45)^2$ and the total force on the base was $\left(9000 \frac{\text{lb}}{\text{ft}^2}\right)(2025\pi \text{ ft}^2) \approx 57,225,526 \text{ lb}$.

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Example The Great Molasses Flood of 1919

(b) Partition the band from depth 89 ft to depth 90 ft into narrower bands of width Δy and choose a depth y_k in each one. The pressure at this depth y_k is $p = wh = 100y_k$ lb/ft^2. The force against each narrow band is approximately

$$\text{pressure} \times \text{area} = (100y_k)(90\pi\Delta y)$$
$$= 9000\pi\, y_k \Delta y \text{ lb.}$$

Add the forces against all bands in the partition and take the limit as the norms go to zero. The force against the bottom foot of tank wall is:

$$F = \int_{89}^{90} 9000\pi\, y\, dy = 9000\pi \int_{89}^{90} y\, dy \approx 2,530,553 \text{ lb.}$$

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Probability Density Function (pdf)

A **probability density function** is a function $f(x)$ with domain all reals such that

$$f(x) \geq 0 \text{ for all } x \text{ and } \int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Then the probability associated with an interval $[a, b]$ is

$$\int_{a}^{b} f(x)\,dx.$$

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Normal Probability Density Function (pdf)

The **normal probability density function (Gaussian curve)** for a population with mean μ and standard deviation σ is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

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The 68-95-99.7 Rule for Normal Distributions

Given a normal curve,

- 68% of the area will lie within σ of the mean μ,
- 95% of the area will lie within 2σ of the mean μ,
- 99.7% of the area will lie within 3σ of the mean μ.

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Example A Telephone Help Line

Suppose a telephone help line takes a mean of 1.5 minutes to answer a call. If the standard deviation is 0.2, then 68% of the calls are answered in the range of 1.3 to 1.7 minutes and 99.7% of the calls are answered in the range of 0.9 to 2.1 minutes.

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Example Weights of Coffee Cans

Suppose that coffee cans marked as "8 ounces" of coffee have a mean weight of 8.2 ounces and a standard deviation of 0.3 ounces.

(a) What percentage of all such cans can be expected to weigh between 8 and 9 ounces?

(b) What percentage would we expect to weigh less than 8 ounces?

(c) What is the probability that a can weighs exactly 8 ounces?

Assume a normal pdf will model these probabilities.

$$f(x) = \frac{1}{0.3\sqrt{2\pi}} e^{-(x-8.2)^2/0.18}$$

(a) Find the area under the curve of $f(x)$ from 8 to 9:

$\text{NINT}(f(x), x, 8, 9) \approx 0.744$

About 74.4% of the cans will have between 8 and 9 ounces.

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Example Weights of Coffee Cans

(b) The graph of $f(x)$ approaches the x-axis as an asymptote.

When $x = 5$, the graph of $f(x)$ is very close to the x-axis.

$\text{NINT}(f(x), x, 5, 8) \approx 0.252$.

Expect about 25.2% of the boxes to weigh less than 8 ounces.

(c) This would be the integral from 8 to 8, which is zero.

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Quick Quiz Sections 8.4 and 8.5

You should solve the following problems without using a graphing calculator.

1. The length of a curve from $x = 0$ to $x = 1$ is given by $\int_0^1 \sqrt{1 + 16x^6}\, dx$.

If the curve contains the point (1,4), which of the following could be an equation for this curve?

(A) $y = x^4 + 3$

(B) $y = x^4 + 1$

(C) $y = 1 + 16x^6$

(D) $y = \sqrt{1 + 16x^6}$

(E) $y = x + \frac{x^7}{7}$

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Quick Quiz Sections 8.4 and 8.5

2. Which of the following gives the length of the path described by the parametric equations $x = \frac{1}{4}t^4$ and $y = t^3$, where $0 \le t \le 2$?

(A) $\int_0^2 t^6 + 9t^4\,dt$

(B) $\int_0^2 \sqrt{t^6 + 1}\,dt$

(C) $\int_0^2 \sqrt{1 - 9t^4}\,dt$

(D) $\int_0^2 \sqrt{t^6 + 9t^4}\,dt$

(E) $\int_0^2 \sqrt{t^3 + 3t^2}\,dt$

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(E) $\int_0^2 \sqrt{t^3 + 3t^2}\,dt$

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Quick Quiz Sections 8.4 and 8.5

3. The base of a solid is a circle of radius 2 inches. Each cross section perpendicular to a certain diameter is a square with one side lying in the circle. The volume of the solid in cubic inches is

(A) 4

(B) 4π

(C) $\frac{32}{3}$

(D) $\frac{32\pi}{3}$

(E) 8π

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Quick Quiz Sections 8.4 and 8.5

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(D) $\frac{32\pi}{3}$

(E) 8π

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Chapter Test

1. A toy car slides down a ramp and coasts to a stop after 5 sec. Its velocity from $t = 0$ to $t = 5$ is modeled by $v(t) = t^2 - 0.2t^3$ ft/sec. How far does it travel? Set up an integral and evaluate it to answer the question.

2. Find the area of the region enclosed by $y = x + 1$ and $y = 3 - x^2$.

3. Find the area of the region enclosed by $y = x^3 - x$ and $y = \dfrac{x}{x^2 + 1}$.

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Chapter Test

4. You drove an 800-gallon tank truck from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You had started out with a full tank of water, had climbed at a steady rate, and had taken 50 minutes to accomplish the 4750-ft elevation change. Assuming that the water leaked out at a steady rate, how much work was spent in carrying the water to the summit? Water weighs 8 lb/gal. (Do not count the work done getting you and your truck to the top.)

5. If a force of 80 N is required to hold a spring 0.3 m beyond its unstressed length, how much work does it take to stretch the spring this far? How much work does it take to stretch the spring an additional meter?

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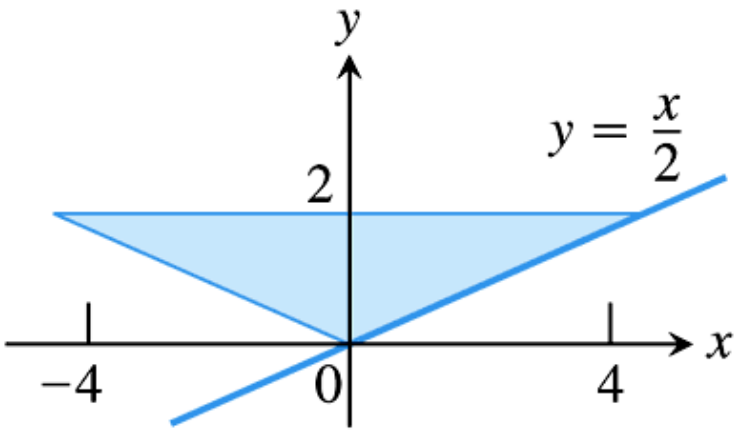
Chapter Test

6. The vertical triangular plate shown below is the end plate of a feeding trough full of hog slop, weighing 80 pounds per cubic foot. What is the force against the plate?



UNITS IN FEET

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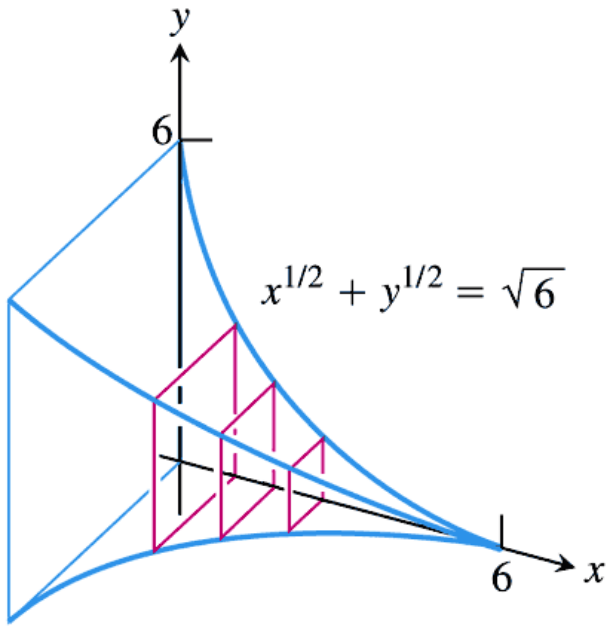
Chapter Test

7. A solid lies between planes perpendicular to the x-axis at $x = 0$ and at $x = 6$. The cross sections between the planes are squares whose bases run from the x-axis up to the curve $\sqrt{x} + \sqrt{y} = \sqrt{6}$. Find the volume of the solid.



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Chapter Test

8. A researcher measures the lengths of 3-year-old yellow perch in a fish hatchery and finds that they have a mean length of 17.2 cm with a standard deviation of 3.4 cm. What proportion of 3-year-old yellow perch raised under similar conditions can be expected to reach a length of 20 cm or more?

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Chapter Test Solutions

1. A toy car slides down a ramp and coasts to a stop after 5 sec. Its velocity from $t = 0$ to $t = 5$ is modeled by $v(t) = t^2 - 0.2t^3$ ft/sec. How far does it travel? Set up an integral and evaluate it to answer the question.

≈ 10.417 ft

2. Find the area of the region enclosed by $y = x + 1$ and $y = 3 - x^2$.

$\frac{9}{2}$

3. Find the area of the region enclosed by $y = x^3 - x$ and $y = \frac{x}{x^2 + 1}$.

≈ 1.2956

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Chapter Test Solutions

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22,800,000 ft-lb

5. If a force of 80 N is required to hold a spring 0.3 m beyond its unstressed length, how much work does it take to stretch the spring this far? How much work does it take to stretch the spring an additional meter?

≈ 213.3 J

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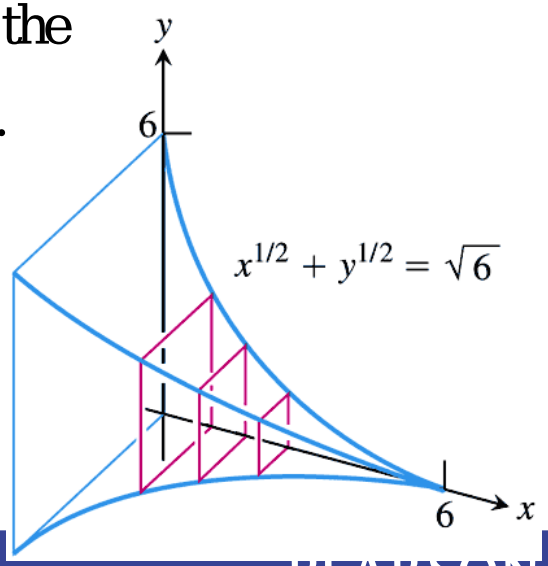
Chapter Test Solutions

6. The vertical triangular plate shown below is the end plate of a feeding trough full of hog slop, weighing 80 pounds per cubic foot. What is the force against the plate?

≈ 426.67 lbs

7. A solid lies between planes perpendicular to the x-axis at $x = 0$ and at $x = 6$. The cross sections between the planes are squares whose bases run from the x-axis up to the curve $\sqrt{x} + \sqrt{y} = \sqrt{6}$. Find the volume of the solid.

≈ 14.4



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Chapter Test Solutions

8. A researcher measures the lengths of 3-year-old yellow perch in a fish hatchery and finds that they have a mean length of 17.2 cm with a standard deviation of 3.4 cm. What proportion of 3-year-old yellow perch raised under similar conditions can be expected to reach a length of 20 cm or more?

≈ 0.2051 (20.5%)

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